

Classical and Quantum Perturbation Theory for two Non-Resonant Oscillators with Quartic Interaction

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Abstract. We study the classical and quantum perturbation theory for two non-resonant oscillators coupled by a nonlinear quartic interaction. In particular we analyze the question of quantum corrections to the torus quantization of the classical perturbation theory (semiclassical mechanics). We obtain up to the second order of perturbation theory an explicit analytical formula for the quantum energy levels, which is the semiclassical one plus quantum corrections. We compare the "exact" quantum levels obtained numerically to the semiclassical levels studying also the effects of quantum corrections.

Key words: Classical Perturbation Theory; Quantum Mechanics; General Mechanics

1 Introduction

Nowadays there is considerable renewed interest in the transition from classical mechanics to quantum mechanics, a powerful motivation behind that being the problem of the so-called quantum chaos [1–3]. An important aspect is represented by the semiclassical quantization formula of the (regular) energy levels for quasi-integrable systems [4–6], the so-called torus quantization, initiated by Einstein [7] and completed by Maslov [8].

It has been recently shown [9,10] that, for perturbed non-resonant harmonic oscillators, the algorithm of classical perturbation theory can be used to formulate the quantum mechanical perturbation theory as the semiclassically quantized classical perturbation theory equipped with the quantum corrections in powers of \hbar "correcting" the classical Hamiltonian that appears in the classical algorithm. In effect, one can explicitly calculate corrections to the Einstein–Brillouin–Keller (EBK) quantization of the classical tori [4]. For example, the quantum corrections of the one-dimensional x^4 perturbed harmonic oscillator have been studied in great details by Alvarez, Graffi and Silverstone [11].

Examples of rather detailed studies of semiclassical approximations, their resummations and of the Birkhoff–Gustavson normal forms can be found in Ali and Wood [12] and Ali, Wood and Devitt [13]. Another example of a rather complete semiclassical analysis of a one-dimensional system, namely the quartic oscillator, has been published by Voros [14].

The aim of this paper is to extend previous studies to a two-dimensional system, which is more interesting because it is nonintegrable and thus generic. The integrable systems are rather exceptional in the sense that they are typically isolated points in the functional space of Hamiltonians and their measure is zero in this space. If we randomly choose a system realized in nature, the probability is one that the system is nonintegrable [15].

2 Classical Perturbation Theory

The model is given by two non-resonant oscillators coupled by a nonlinear quartic interaction of strength g (Pullen and Edmonds [16]):

$$H = \frac{\omega_1}{2}(p_1^2 + q_1^2) + \frac{\omega_2}{2}(p_2^2 + q_2^2) + gq_1^2q_2^2. \quad (1)$$

Note that the same Hamiltonian has been obtained for the Yang–Mills–Higgs classical mechanics. The non–integrability of this system and its transition from order to chaos has been studied in classical with Poincare sections and in quantum mechanics with the spacing distribution of energy levels [17,18].

Through the canonical transformation in action–angle variables [19,20]:

$$q_k = \sqrt{2I_k} \cos \theta_k, \quad p_k = \sqrt{2I_k} \sin \theta_k, \quad k = 1, 2 \quad (2)$$

the Hamiltonian can be written:

$$H = H_0(I_1, I_2) + gV(I_1, I_2, \theta_1, \theta_2), \quad (3)$$

where:

$$H_0(I_1, I_2) = \omega_1 I_1 + \omega_2 I_2, \quad (4)$$

$$V(I_1, I_2, \theta_1, \theta_2) = 4I_1 I_2 \cos^2 \theta_1 \cos^2 \theta_2. \quad (5)$$

Following the classical perturbation theory [19,20], we search for a canonical transformation $(I_1, I_2, \theta_1, \theta_2) \rightarrow (\tilde{I}_1, \tilde{I}_2, \tilde{\theta}_1, \tilde{\theta}_2)$ to obtain a new Hamiltonian that depends only on the new action variables up to the second order in a power series of g :

$$\tilde{H}(\tilde{I}_1, \tilde{I}_2) = \tilde{H}_0(\tilde{I}_1, \tilde{I}_2) + g\tilde{H}_1(\tilde{I}_1, \tilde{I}_2) + g^2\tilde{H}_2(\tilde{I}_1, \tilde{I}_2). \quad (6)$$

The generator S of the canonical transformation is supposed to be capable of expansion as a power series in g of the form:

$$S(\tilde{I}_1, \tilde{I}_2, \theta_1, \theta_2) = \tilde{I}_1 \theta_1 + \tilde{I}_2 \theta_2 + gS_1(\tilde{I}_1, \tilde{I}_2, \theta_1, \theta_2) + g^2S_2(\tilde{I}_1, \tilde{I}_2, \theta_1, \theta_2), \quad (7)$$

and to satisfy the equations:

$$I_k = \frac{\partial S}{\partial \theta_k} = \tilde{I}_k + g \frac{\partial S_1}{\partial \theta_k} + g^2 \frac{\partial S_2}{\partial \theta_k}, \quad (8)$$

$$\tilde{\theta}_k = \frac{\partial S}{\partial \tilde{I}_k} = \theta_k + g \frac{\partial S_1}{\partial \tilde{I}_k} + g^2 \frac{\partial S_2}{\partial \tilde{I}_k}. \quad (9)$$

From the Hamilton–Jacobi equation:

$$H_0\left(\frac{\partial S}{\partial \theta_1}, \frac{\partial S}{\partial \theta_2}\right) + gV\left(\frac{\partial S}{\partial \theta_1}, \frac{\partial S}{\partial \theta_2}, \theta_1, \theta_2\right) = \tilde{H}_0(\tilde{I}_1, \tilde{I}_2) + g\tilde{H}_1(\tilde{I}_1, \tilde{I}_2) + g^2\tilde{H}_2(\tilde{I}_1, \tilde{I}_2), \quad (10)$$

we have a number of differential equations that result on equating the coefficients of the powers of g :

$$\tilde{H}_0(\tilde{I}_1, \tilde{I}_2) = H_0(\tilde{I}_1, \tilde{I}_2) = \omega_1 \tilde{I}_1 + \omega_2 \tilde{I}_2, \quad (11)$$

$$\tilde{H}_1(\tilde{I}_1, \tilde{I}_2) = \left(\omega_1 \frac{\partial S_1}{\partial \theta_1} + \omega_2 \frac{\partial S_1}{\partial \theta_2}\right) + V(\tilde{I}_1, \tilde{I}_2, \theta_1, \theta_2), \quad (12)$$

$$\tilde{H}_2(\tilde{I}_1, \tilde{I}_2) = \left(\omega_1 \frac{\partial S_2}{\partial \theta_1} + \omega_2 \frac{\partial S_2}{\partial \theta_2}\right) + \left(\frac{\partial V}{\partial I_1} \frac{\partial S_1}{\partial \theta_1} + \frac{\partial V}{\partial I_2} \frac{\partial S_1}{\partial \theta_2}\right). \quad (13)$$

The unknown functions \tilde{H}_1 , S_1 , \tilde{H}_2 and S_2 may be determined by averaging over the time variation of the unperturbed motion.

At the first order in g we obtain:

$$\tilde{H}_1(\tilde{I}_1, \tilde{I}_2) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 V(\tilde{I}_1, \tilde{I}_2, \theta_1, \theta_2) = \tilde{I}_1 \tilde{I}_2, \quad (14)$$

and

$$\begin{aligned} S_1(\tilde{I}_1, \tilde{I}_2, \theta_1, \theta_2) = & -\frac{1}{4} \tilde{I}_1 \tilde{I}_2 \left[\frac{2}{\omega_1} \sin 2\theta_1 + \frac{2}{\omega_2} \sin 2\theta_2 + \right. \\ & \left. + \frac{1}{\omega_1 - \omega_2} \sin 2(\theta_1 - \theta_2) + \frac{1}{\omega_1 + \omega_2} \sin 2(\theta_1 + \theta_2) \right]. \end{aligned} \quad (15)$$

At the second order in g we have:

$$\tilde{H}_2(\tilde{I}_1, \tilde{I}_2) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \left(\frac{\partial V}{\partial I_1} \frac{\partial S_1}{\partial \theta_1} + \frac{\partial V}{\partial I_2} \frac{\partial S_1}{\partial \theta_2} \right)$$

$$= -\frac{1}{8}\tilde{I}_1\tilde{I}_2[4(\frac{\tilde{I}_1}{\omega_2} + \frac{\tilde{I}_2}{\omega_1}) - \frac{(\tilde{I}_1 - \tilde{I}_2)}{\omega_1 - \omega_2} + \frac{(\tilde{I}_1 + \tilde{I}_2)}{\omega_1 + \omega_2}]. \quad (16)$$

We observe that the integrable approximate Hamiltonian (6) could be obtained alternatively as the Birkhoff–Gustavson normal form, which is a purely algebraic method of calculating the action variables order by order for perturbed harmonic oscillators with polynomial perturbations [21]. The Hamiltonian (6) depends only on the actions so that by an application of the EBK rule:

$$\tilde{I}_1 = (n_1 + \frac{1}{2})\hbar, \quad \tilde{I}_2 = (n_2 + \frac{1}{2})\hbar, \quad (17)$$

to equations (11), (14) and (16), we obtain a semiclassical analytical formula of the energy levels. Obviously \hbar is the Planck constant and n_1, n_2 are two integer quantum number.

Note that the torus quantization can be applied because the normal form obtained (equation 6) is integrable. The new action variables remain very close to their initial conditions. This is no longer true, in general, for higher dimensions (see [19,20]). In the next section we show how to connect our semiclassical formula with the usual quantum perturbation theory.

3 Quantum Perturbation Theory

In quantum mechanics the generalized coordinates are operators which satisfy the usual commutation rules for Bosons $[\hat{q}_k, \hat{p}_l] = \hat{q}_k\hat{p}_l - \hat{p}_l\hat{q}_k = i\hbar\delta_{kl}$, with $k, l = 1, 2$. Introducing the creation and destruction operators:

$$\hat{a}_k = \frac{1}{\sqrt{2\hbar}}(\hat{q}_k + i\hat{p}_k), \quad \hat{a}_k^+ = \frac{1}{\sqrt{2\hbar}}(\hat{q}_k - i\hat{p}_k), \quad (18)$$

the quantum Hamiltonian can be written:

$$\hat{H} = \hat{H}_0 + g\hat{V}, \quad (19)$$

where:

$$\hat{H}_0 = \hbar\omega_1(\hat{a}_1^+\hat{a}_1 + \frac{1}{2}) + \hbar\omega_2(\hat{a}_2^+\hat{a}_2 + \frac{1}{2}), \quad (20)$$

$$\hat{V} = \frac{\hbar^2}{4}(\hat{a}_1 + \hat{a}_1^+)^2(\hat{a}_2 + \hat{a}_2^+)^2. \quad (21)$$

If $|n_1 n_2\rangle$ is the basis of the occupation numbers of the two harmonic oscillators, the matrix elements are:

$$\langle n_1' n_2' | \hat{H}_0 | n_1 n_2 \rangle = \hbar[\omega_1(n_1 + \frac{1}{2}) + \omega_2(n_2 + \frac{1}{2})]\delta_{n_1' n_1} \delta_{n_2' n_2}, \quad (22)$$

and:

$$\begin{aligned} \langle n_1' n_2' | \hat{V} | n_1 n_2 \rangle &= \frac{\hbar^2}{4} [\sqrt{n_1(n_1-1)}\delta_{n_1' n_1-2} + \sqrt{(n_1+1)(n_1+2)}\delta_{n_1' n_1+2} + (2n_1+1)\delta_{n_1' n_1}] \times \\ &\times [\sqrt{n_2(n_2-1)}\delta_{n_2' n_2-2} + \sqrt{(n_2+1)(n_2+2)}\delta_{n_2' n_2+2} + (2n_2+1)\delta_{n_2' n_2}]. \end{aligned} \quad (23)$$

The Rayleigh–Schrödinger perturbation theory [22] up to the second order gives us:

$$E(n_1\hbar, n_2\hbar) = E_0(n_1\hbar, n_2\hbar) + gE_1(n_1\hbar, n_2\hbar) + g^2E_2(n_1\hbar, n_2\hbar), \quad (24)$$

where:

$$E_0(n_1\hbar, n_2\hbar) = \hbar[\omega_1(n_1 + \frac{1}{2}) + \omega_2(n_2 + \frac{1}{2})], \quad (25)$$

$$E_1(n_1\hbar, n_2\hbar) = \langle n_1 n_2 | \hat{V} | n_1 n_2 \rangle, \quad (26)$$

$$E_2(n_1\hbar, n_2\hbar) = \sum_{\substack{n_1' n_2' \\ (n_1', n_2') \neq (n_1, n_2)}} \frac{|\langle n_1' n_2' | \hat{V} | n_1 n_2 \rangle|^2}{\hbar[\omega_1(n_1 - n_1') + \omega_2(n_2 - n_2')]}. \quad (27)$$

We obtain immediately:

$$E_1(n_1\hbar, n_2\hbar) = \hbar^2(n_1 + \frac{1}{2})(n_2 + \frac{1}{2}), \quad (28)$$

and after some calculations:

$$\begin{aligned} E_2(n_1\hbar, n_2\hbar) = & \frac{\hbar^3}{32} \left[\frac{n_1(n_1-1)n_2(n_2-1)}{\omega_1 + \omega_2} - \frac{(n_1+1)(n_1+2)(n_2+1)(n_2+2)}{\omega_1 + \omega_2} + \right. \\ & + \frac{n_1(n_1-1)(n_2+1)(n_2+2)}{\omega_1 - \omega_2} - \frac{(n_1+1)(n_1+2)n_2(n_2-1)}{\omega_1 - \omega_2} + \\ & + \frac{n_1(n_1-1)(2n_2+1)^2}{\omega_1} - \frac{(n_1+1)(n_1+2)(2n_2+1)^2}{\omega_1} + \\ & \left. + \frac{(2n_1+1)^2n_2(n_2-1)}{\omega_2} - \frac{(2n_1+1)^2(n_2+1)(n_2+2)}{\omega_2} \right]. \quad (29) \end{aligned}$$

The zero and first order quantum terms coincide with the semi-classical ones:

$$E_0(n_1\hbar, n_2\hbar) = \tilde{H}_0((n_1 + \frac{1}{2})\hbar, (n_2 + \frac{1}{2})\hbar), \quad (30)$$

$$E_1(n_1\hbar, n_2\hbar) = \tilde{H}_1((n_1 + \frac{1}{2})\hbar, (n_2 + \frac{1}{2})\hbar), \quad (31)$$

and the second order quantum term can be written:

$$E_2(n_1\hbar, n_2\hbar) = \tilde{H}_2((n_1 + \frac{1}{2})\hbar, (n_2 + \frac{1}{2})\hbar) + \hbar^2 Q_2((n_1 + \frac{1}{2})\hbar, (n_2 + \frac{1}{2})\hbar), \quad (32)$$

where:

$$Q_2((n_1 + \frac{1}{2})\hbar, (n_2 + \frac{1}{2})\hbar) = -\frac{3}{32} \left[\frac{(n_1 - n_2)\hbar}{\omega_1 - \omega_2} + \frac{(n_1 + n_2 + 1)\hbar}{\omega_1 + \omega_2} \right]. \quad (33)$$

The quantum series rearranges directly into the classical canonical perturbation series plus quantum corrections proportional to successive powers of \hbar [10].

The term $\hbar^2 Q_2$ represents the quantum corrections to the EBK quantization up to the second order of perturbation theory. These quantum corrections depend linearly on quantum numbers. For a quite similar system, Robnik [21] obtained that the spectra differ only by an additive constant independent of the quantum numbers (actions), but he studied only the first order of perturbation theory.

To conclude this section we mention the general problem of quantization: Only the quantization of coordinate space or the quantization of linear canonical transformations of coordinate space yields the right quantum mechanics, whose results agree with experiments. This is exactly the approach implemented in our present case (18), and thus our quantization is equivalent to the coordinate space quantization.

4 Numerical Results

We compute the energy levels with a numerical diagonalization of the truncated matrix of the Hamiltonian (19) in the basis of the unperturbed oscillators (see [23] for a more technical discussion). The numerical energy levels depend on the dimension of the truncated matrix: We compute the numerical levels in double precision increasing the matrix dimension until the first 100 levels converge within 8 digits (matrix dimension 1225×1225). This means that the agreement between numerical values and real ones is very good and we use the word "exact" to indicate the numerically computed energy levels.

In Table 1 we compare the "exact" levels to the levels of the quantum perturbation theory and to the semiclassical results. A very good agreement is observed for the lowest energy levels.

In table 2 we show the error in units of the mean level spacing D between the "exact" levels and the levels obtained with semiclassical and quantum perturbation theory. We observe that the algorithm provided by the appropriate semiclassical quantization is comparable to the algorithm provided by ordinary quantum perturbation theory but the quantal corrections do not always increase the accuracy. Thus for some of the calculated levels the semiclassical quantization gives better results than the quantum perturbation theory.

In Table 3 we show the error, in units of the mean level spacing D , between the "exact" levels and the semiclassical levels. As is seen, by decreasing \hbar from 1 to 10^{-1} the quality of the approximation improves considerably. However, it is important to observe that if \hbar , no matter how small, is kept fixed, the EBK quantization (torus quantization) of the individual levels is only a first order approximation of an expansion in \hbar . Therefore, *in general* the accuracy of the approximation decreases for higher levels. To get a good agreement it is necessary, as is well known, to implement the classical limit, i.e. $\hbar \rightarrow 0$ and $n_1, n_2 \rightarrow \infty$, while at the same time keeping the actions $\tilde{I}_1 = (n_1 + 1/2)\hbar$ and $\tilde{I}_2 = (n_2 + 1/2)\hbar$ constant [23].

5 Conclusions

We have examined the transition between the classical and the quantum mechanics for a two-dimensional, nonintegrable and non-resonant system. Up to the second order of perturbation theory we have decomposed the quantum description into the classical description (i.e. the leading semiclassical term) plus quantum corrections which depend linearly on the quantum numbers. The semiclassical energy levels and the levels obtained with quantum perturbation theory are in good agreement with the "exact" numerical ones: The semiclassical quantization is comparable to the quantum perturbation theory, and for some levels the semiclassical quantization gives better results than quantum perturbation theory.

There are two series involved in the work. One in powers of \hbar because of the semiclassical approximation and another in the interaction parameter g . The first one is an asymptotic expansion (see some very recent results in [24–27]) but the second one may be not convergent [28]. The classical and quantum perturbation series typically diverge and thus do not necessarily describe the exact levels, not even after a certain resummation (except for some important notable exceptions like the anharmonic oscillators with f degrees of freedom with a polynomial perturbing potential which is asymptotically positive definite [29,30]), it is important to compare the semiclassical approximation (and the quantal perturbation results) with the exact spectra, which in general is impossible, since we generally do not have explicit solutions of the Schrödinger problem in a closed form. Therefore, we stress the importance of specific case studies like the present one, in order to get a better understanding of the quality of semiclassical mechanics.

Finally we note that the extraction of quantum corrections for resonant systems is a more intricate procedure; some initial results for perturbed resonant oscillators can be found in Graffi [31].

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E^{ex}	$E_{n_1 n_2}^{sc}$	$E_{n_1 n_2}^{qp}$	(n_1, n_2)
1.230722	1.230990	1.230522	(0,0)
2.275974	2.273214	2.274701	(1,0)
2.689415	2.690856	2.687816	(0,1)
3.316524	3.308447	3.311808	(2,0)
3.820434	3.814018	3.812833	(1,1)
4.146646	4.148302	4.142610	(0,2)
4.354307	4.336609	4.341846	(3,0)
4.937708	4.915967	4.916677	(2,1)
5.359848	5.347322	5.345305	(1,2)
5.390110	5.357700	5.364811	(4,0)
5.603778	5.603248	5.594904	(0,3)
6.047742	5.996702	5.999287	(3,1)
6.424398	6.371719	6.380706	(5,0)
6.546966	6.510986	6.509044	(2,2)
6.897049	6.873125	6.866657	(1,3)
7.062932	7.055694	7.044699	(0,4)
7.152476	7.056224	7.060684	(4,1)
7.457506	7.378668	7.389530	(6,0)
7.723943	7.639295	7.639228	(3,2)
8.144146	8.093505	8.088912	(2,3)

Table 1: Comparison between "exact" levels and levels obtained by perturbation theories. First 20 levels. E^{ex} are "exact" levels, $E_{n_1 n_2}^{sc}$ are semiclassical levels, and $E_{n_1 n_2}^{qp}$ are levels obtained with quantum perturbation theory, where n_1 and n_2 are the quantum number. $\hbar = 1$, $g = 10^{-1}$, $\omega_1 = 1$ and $\omega_2 = \sqrt{2}$.

$ E^{ex} - E_{n_1 n_2}^{sc} /D$	$ E^{ex} - E_{n_1 n_2}^{qp} /D$	(n_1, n_2)
1.0611359 $\cdot 10^{-3}$	1.1284242 $\cdot 10^{-3}$	(0,0)
1.5578579 $\cdot 10^{-2}$	7.1859419 $\cdot 10^{-3}$	(1,0)
8.1338054 $\cdot 10^{-3}$	9.0260478 $\cdot 10^{-3}$	(0,1)
4.5591835 $\cdot 10^{-2}$	2.6613854 $\cdot 10^{-2}$	(2,0)
3.6215890 $\cdot 10^{-2}$	4.2791300 $\cdot 10^{-2}$	(1,1)
9.3476856 $\cdot 10^{-3}$	2.2781115 $\cdot 10^{-2}$	(0,2)
9.9898852 $\cdot 10^{-2}$	7.0337765 $\cdot 10^{-2}$	(3,0)
0.1227176	0.1187100	(2,1)
7.0703819 $\cdot 10^{-2}$	9.2249520 $\cdot 10^{-2}$	(1,2)
0.1829406	0.1428019	(4,0)
2.9902905 $\cdot 10^{-2}$	5.0089385 $\cdot 10^{-2}$	(0,3)
0.2880960	0.2735053	(3,1)
0.2973495	0.2466222	(5,0)
0.2030921	0.2140520	(2,2)
0.1350395	0.1715501	(1,3)
4.0854741 $\cdot 10^{-2}$	0.1029161	(0,4)
0.5432989	0.5181223	(4,1)
0.4450069	0.3836938	(6,0)
0.4778005	0.4781800	(3,2)
0.2858459	0.3117708	(2,3)

Table 2: The error measured in units of the mean level spacing D for the first 20 levels. D is calculated for the lowest 100 levels. E^{ex} are "exact" levels, $E_{n_1 n_2}^{sc}$ are semiclassical levels, and $E_{n_1 n_2}^{qp}$ are levels obtained with quantum perturbation theory, where n_1 and n_2 are the quantum number. $\hbar = 1$, $g = 10^{-1}$, $\omega_1 = 1$ and $\omega_2 = \sqrt{2}$.

$\hbar = 1$	$\hbar = 10^{-1}$	(n_1, n_2)
$1.0611359 \cdot 10^{-3}$	$2.4773894 \cdot 10^{-5}$	(0,0)
$1.5578579 \cdot 10^{-2}$	$1.0003044 \cdot 10^{-4}$	(1,0)
$8.1338054 \cdot 10^{-3}$	$1.8136360 \cdot 10^{-4}$	(0,1)
$4.5591835 \cdot 10^{-2}$	$2.5054353 \cdot 10^{-4}$	(2,0)
$3.6215890 \cdot 10^{-2}$	$5.6091835 \cdot 10^{-6}$	(1,1)
$9.3476856 \cdot 10^{-3}$	$3.1972348 \cdot 10^{-4}$	(0,2)
$9.9898852 \cdot 10^{-2}$	$4.3938606 \cdot 10^{-4}$	(3,0)
0.1227176	$2.3371598 \cdot 10^{-4}$	(2,1)
$7.0703819 \cdot 10^{-2}$	$9.3486393 \cdot 10^{-5}$	(1,2)
0.1829406	$7.0301769 \cdot 10^{-4}$	(4,0)
$2.9902905 \cdot 10^{-3}$	$4.4125578 \cdot 10^{-4}$	(0,3)
0.2880960	$6.3570746 \cdot 10^{-4}$	(3,1)
0.2973495	$2.3932516 \cdot 10^{-4}$	(5,0)
0.2030921	$1.0582660 \cdot 10^{-3}$	(2,2)
0.1350395	$1.1966258 \cdot 10^{-4}$	(1,3)
$4.0854741 \cdot 10^{-2}$	$1.2452388 \cdot 10^{-3}$	(0,4)
0.5432989	$5.2726327 \cdot 10^{-4}$	(4,1)
0.4450069	$8.1146188 \cdot 10^{-4}$	(6,0)
0.4778005	$1.5294374 \cdot 10^{-3}$	(3,2)
0.2858459	$3.0663537 \cdot 10^{-4}$	(2,3)

Table 3: The error measured in units of the mean level spacing D between "exact" levels and semiclassical levels. First 20 levels. D is calculated for the lowest 100 levels. $g = 10^{-1}$, $\omega_1 = 1$ and $\omega_2 = \sqrt{2}$.

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